Three Dimensional Manifolds All of Whose Geodesics Are Closed

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ABSTRACT

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We present some results concerning the Morse Theory of the energy function on the free loop space of S^3 for metrics all of whose geodesics are closed. We also show how these results may be regarded as partial results on the Berger Conjecture in dimension three.

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Chapter 1

Introduction

About 30 years ago Berger conjectured that on a simply connected manifold all of whose geodesics are closed, all geodesics have the same least period. In addition to the spheres and projective spaces with the standard metrics, the so-called Zoll metrics on S^n have this property as well; see [Bes78, Corollary 4.16]. The weaker statement that there exists a common period is a special case of a theorem due to Wadsley; see [Bes78, Theorem 7.12]. The lens spaces with the canonical metrics show that simply connectedness is necessary. On $S^{2n+1}/\mathbb{Z}_k$, $k > 2$ all geodesics are closed with common period 2π, but there exist geodesics of smaller period.

Bott and Samelson studied the topology of such manifolds and showed that they must have the same cohomology ring as a compact rank one symmetric space. In 1982 Gromoll and Grove proved the Berger Conjecture for metrics on S^2, [GG82, Theorem 1].

In this thesis we present some results on the Morse theory on the free loop space of S^3 for metrics all of whose geodesics are closed. We also see how these results may be regarded as partial results on the Berger Conjecture.

Before we state the results we will review some basic notions from Morse theory on the free loop space of a Riemannian manifold; see Chapter 2 for an introduction.

Let M be a Riemannian manifold and let the free loop space ΛM be the set of absolutely continuous maps $c\colon S^1 \to M$ with square integrable derivative and let the energy function $\mathrm{E}\colon \Lambda M \to \mathbb{R}$ be given by $\mathrm{E}(c) = \int_0^{2\pi} |\dot{c}(t)|^2 dt$. The free loop space ΛM can be given the structure of a smooth Hilbert manifold which makes E into a smooth function. It follows from the first variation formula that the critical points of E are the closed geodesics on M. The group $O(2)$ acts on the free loop space by reparameterization and since the action leaves E invariant, a critical point is never isolated. If the critical sets are submanifolds of ΛM, then we say that such a manifold N is nondegenerate if the null space of the Hessian $\mathrm{Hess}_c(\mathrm{E})$ is equal to the tangent space $T_c N$. If all critical manifolds are nondegenerate in this sense we say that E is a Morse-Bott function.

Let $\Lambda^a M$ be the set $\mathrm{E}^{-1}([0,a]) \subseteq \Lambda M$. If N is the only critical submanifold of energy a, one can use the gradient flow to show that there is a homotopy equivalence $\Lambda^{a+\epsilon} M \simeq \Lambda^{a-\epsilon} M \cup_f D(\xi^-)$ for some gluing map $f\colon S(\xi^-) \to \Lambda^{a-\epsilon} M$, where ξ^- is the negative bundle over N whose fiber consists of the sum of the negative eigenspaces of $\mathrm{Hess}_c(\mathrm{E})$. The rank of ξ^- is denoted $\lambda(N)$ and is called the index of the critical manifold. The spaces $D(\xi^-)$ and $S(\xi^-)$ are the disk respectively sphere bundle of ξ^-. Excision gives an isomorphism $H^i(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M; R) \cong H^i(D(\xi^-), S(\xi^-); R)$ and if the negative bundle over N is orientable the Thom isomorphism yields $H^i(D(\xi^-), S(\xi^-); R) \cong H^{i-\lambda(N)}(N; R)$ for any coefficient ring R. If N is not orientable the last isomorphism holds with $\mathbb{Z}_2$ coefficients. If R is a field we say that E is perfect if $H^j(\Lambda M; R) = \bigoplus_j H^{j-\lambda(N_j)}(N_j; R)$, where the sum is over all critical manifolds.

For a topological group G we let EG be a contractible topological space on which G acts freely and let $EG/G = BG$ be the classifying space of G. For a G-space X the G-equivariant cohomology of X is defined to be the usual cohomology of the quotient space $(X \times EG)/G = X_G$. While the action of G on X might not be free, the diagonal action of G on $X \times EG$ is always free, and the equivariant cohomology of X models the cohomology of X/G is the sense that for a free action we have $H_G^*(X; R) \cong H^*(X/G; R)$. The negative bundles are $O(2)$-vector bundles and one can for any group $G \subseteq O(2)$ do equivariant Morse Theory analogously to the ordinary theory. In particular, one gets the isomorphism $H_G^i(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M; R) \cong H_G^i(D(\xi^-), S(\xi^-); R)$, and if the G-equivariant negative bundle is orientable, the Thom isomorphism yields $H_G^i(D(\xi^-), S(\xi^-) \cong H_G^{i-\lambda(N)}(N; R)$ for any ring R. If R is a field, we say that E is perfect with respect to G-equivariant cohomology with coefficients in R if $H_G^j(\Lambda M; R) = \bigoplus_j H_G^{j-\lambda(N_j)}(N_j; R)$, where the sum is over all critical manifolds.

We can now state the main results of the thesis.

Theorem 1. *Let g be a metric on S^3 all of whose geodesics are closed. Then relative to the point curves the energy function* E *is perfect with respect to rational S^1-equivariant cohomology.*

The fact that the energy function is a Morse-Bott function was observed by Wiking; see [Wil01, Proof of Step 3]. This allows us to do Morse Theory on ΛS^3. The next theorem allows us to use arbitrary coefficients for the homology.

Theorem 2. *Let g be a metric on S^3 all of whose geodesics are closed. Then the negative bundles over the critical manifolds are orientable both as ordinary and S^1-equivariant vector bundles.*

We have the following structure result for the critical manifolds of the energy function.

Theorem 3. *Let g be a metric on S^3 all of whose geodesics are closed. A critical manifold in ΛS^3 is either diffeomorphic to the unit tangent bundle of S^3 or has the integral cohomology ring of a three dimensional lens space $S^3/\mathbb{Z}_{2k}$.*

The three theorems together imply the following theorem; see Chapter 7, Theorem 3.

Theorem 4. *Let g be a metric on S^3 all of whose geodesics are closed. The geodesics have the same least period if and only if the energy function is perfect for ordinary cohomology.*

One possible way to attack the Berger Conjecture in dimension three is to assume that there exist exceptional closed geodesics of period $2\pi/n$ for $n > 1$ and then try to derive a contradiction by using the results above together with the Bott Iteration Formula, which calculate the index of an iterated geodesic.

We show that there exists only one critical manifold N of index two and that this critical manifold must be three dimensional. By the Bott Iteration Formula we see that for $c \in N$, the index of c^2 is either four or six. If we assume that the sectional curvature K satisfies $1/4 \leq K \leq 1$ we are able to get a contradiction in the case where $\mathrm{Index}(c^2) = 4$ by using a result from [BTZ83]. We have not yet been able to handle the case where $\mathrm{Index}(c^2) = 6$; see Chapter 7 Theorem 4 for a more detailed description of further properties we are able to prove.

Chapter 2

Preliminaries

We will review some basic notions from Morse theory on the free loop space of a Riemannian manifold. The standard reference is [Kli78]; see also [BO04, Chapter 3] and [Hin84] for a short introduction.

Let M be a Riemannian manifold and let $\Lambda M = W^{1,2}(S^1, M)$ be the free loop space of M. The free loop space ΛM can be given the structure of a smooth Hilbert manifold which makes E into a smooth function; see [Kli78, Theorem 1.2.9]. Note that there is a natural inclusion $W^{1,2}(S^1, M) \to C^0(S^1, M)$ and a fundamental theorem states that this map is a homotopy equivalence; see [Kli78, Theorem 1.2.10]. The tangent space at a point $c \in \Lambda M$, $T_c\Lambda M$, consists of all vector fields along c of class $W^{1,2}$, and is a real Hilbert space with inner product given by

$$\langle\langle X, Y\rangle\rangle_1 = \int_0^{2\pi} \langle X(t), Y(t)\rangle + \langle \frac{DX}{dt}, \frac{DY}{dt}\rangle dt,$$

where $\frac{D}{dt}$ is the covariant derivative along c induced by the Levi-Civita connection on M. Let the energy function $\mathrm{E}\colon \Lambda M \to \mathbb{R}$ be given by $\mathrm{E}(c) = \int_0^{2\pi} |\dot{c}(t)|^2 dt$. It follows from the first variation formula that the critical points of E are the closed geodesics on M.

Let N be a critical manifold of E in ΛM and let $c \in N$ be a critical point. We use the convention that the critical sets are maximal and connected throughout. By definition N satisfies the Bott nondegeneracy condition if $T_c N = \mathrm{Ker}(\mathrm{Hess}_c(\mathrm{E}))$. If all critical manifolds are nondegenerate in this sense, the energy function is called a Morse-Bott function. What this says geometrically is that the dimension of the space of periodic Jacobi vector fields along the geodesic c is equal to the dimension of the critical manifold.

Now assume that N_j, $j = 1, \ldots, l$, are the critical manifolds of energy a, that they satisfy the Bott nondegeneracy condition, and that there are no other critical values in the interval $[a-\epsilon, a+\epsilon]$, $\epsilon > 0$. The metric on $T_c\Lambda M$ induces a splitting of the normal bundle, ξ, of N_j in ΛM into a positive and a negative bundle, $\xi = \xi^+ \oplus \xi^-$, such that the Hessian of the energy function is positive definite on ξ^+ and negative definite on ξ^-. Furthermore $\lambda(N_j) = \operatorname{rank} \xi^-$ is called the index of N_j and is finite. Note that the index is constant on each critical manifold, since it is connected. We denote by $\Lambda^a M$ the set $\mathrm{E}^{-1}([0,a]) \subseteq \Lambda M$. Let $D(\xi^-(N_j)) = D(N_j)$ and $S(\xi^-(N_j)) = S(N_j)$ be the disc, respectively sphere, bundle of the negative bundle ξ^- over N_j. Using the gradient flow, one shows that there exists a homotopy equivalence $\Lambda^{a+\epsilon} M \simeq \Lambda^{a-\epsilon} M \cup_f \cup_{j=1}^{l} D(N_j)$ for some gluing maps $f_j\colon S(N_j) \to$

$\Lambda^{a-\epsilon}M$; see [Kli78, Theorem 2.4.10].

If the negative bundle over N_j is orientable for all j, excision and the Thom isomorphism yield

$$H^i(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M; R) \cong \bigoplus_{j=1}^{l} H^i(D(N_j), S(N_j); R) \cong \bigoplus_{j=1}^{l} H^{i-\lambda(N_j)}(N_j; R)$$

for any coefficient ring R. If N is not orientable the isomorphism holds with $\mathbb{Z}_2$ coefficients. The cohomology of $\Lambda^{a+\epsilon}M$ is now determined by the cohomology of $\Lambda^{a-\epsilon}M$ and N_j by the long exact cohomology sequence for the pair $(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M)$. This way one can in principle inductively calculate the cohomology of ΛM from the cohomology of the critical manifolds. If the map $\Lambda^{a+\epsilon}M \to (\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M)$ induces an injective map $H^i(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M; R) \to H^i(\Lambda^{a+\epsilon}M; R)$ for all i, we say that all relative classes can be completed to absolute classes. This is equivalent to all the boundary maps in the long exact cohomology sequence for the pair $(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M)$ being zero. If this holds for all i and all critical values a, we say that E is perfect. If R is a field, perfectness implies that $H^j(\Lambda M; R) = \bigoplus_j H^{j-\lambda(N_j)}(N_j; R)$, where the sum is over all critical manifolds.

For a topological group G we let EG be a contractible topological space on which G acts freely and let $EG/G = BG$ be the classifying space of G. For a G-space X the quotient space $(X \times EG)/G = X \times_G EG = X_G$ is called the Borel construction. The G-equivariant cohomology of X is defined to be the usual cohomology of the Borel construction X_G. We have a fibration $X \to X \times_G EG \to BG$ and, if the G-action on X is free, a fibration $X \times_G EG \to X/G$ with fiber EG, i.e. the map is a (weak) homotopy equivalence. If G acts on a manifold X with finite isotropy groups, the map $X \times_G EG \to X/G$ is a rational homotopy equivalence and the cohomology of X/G is concentrated in finitely many degrees when X is finite dimensional and compact.

For G a group acting on ΛM we consider G-equivariant Morse theory. Let $\xi \to X$ be a vector bundle where G acts on ξ such that $p\colon \xi \to X$ is equivariant and the action is linear on the fibers. For such vector bundles we define the G-vector bundle by $\xi_G = \xi \times_G EG \to X_G$. The G-vector bundle ξ_G is orientable if and only if ξ is orientable and G acts orientation preserving on the fibers. Thus if G is connected ξ_G is orientable if and only if ξ is orientable. For an oriented rank k G-vector bundle ξ_G over X there is a G-equivariant Thom isomorphism, $H^*_G(D((X)), S((X)); R) \cong H^{*-k}_G(X; R)$.

The action of $O(2)$ on S^1 induces an action of $O(2)$ on the free loop space via reparametrization. Since the energy function is invariant under the action of $O(2)$ and $O(2)$ acts by isometries with respect to the inner product used to define the negative bundles, the negative bundles are $O(2)$-bundles in this sense. For any group $G \subseteq O(2)$ we have similarly to the above

$$H^i_G(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M; R) \cong \bigoplus_{j=1}^{l} H^i_G(D(N_j), S(N_j); R) \cong \bigoplus_{j=1}^{l} H_G^{i-\lambda(N_j)}(N_j; R)$$

where as above $D(N_j)$ and $S(N_j)$ are the disk and sphere bundle of the negative bundles ξ^- over the critical manifold N_j $j = 1, \ldots, l$ and where a is the energy of the critical manifolds N_j. The second isomorphism holds for any coefficient ring R if the negative bundle is oriented and with $\mathbb{Z}_2$ coefficients if it is nonorientable. The G-equivariant cohomology of $\Lambda^{a+\epsilon}M$ is determined by the G-equivariant cohomology of $\Lambda^{a-\epsilon}M$ and N_j by the long exact G-equivariant cohomology sequence for the pair $(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M)$. In principle this allows us to inductively calculate the G-equivariant cohomology of ΛM from the G-equivariant cohomology of the critical manifolds. Again we say that E is perfect with respect to G-equivariant cohomology if the map $\Lambda^{a+\epsilon}M \to (\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M)$ induces an injective map $H^i_G(\Lambda^{a+\epsilon}M, \Lambda^{a-\epsilon}M; R) \to H^i_G(\Lambda^{a+\epsilon}M; R)$ for all i and for all critical values a. If R is a field, perfectness implies that $H^j_G(\Lambda M; R) = \bigoplus_j H_G^{j-\lambda(N_j)}(N_j; R)$, where the sum is over all critical manifolds. See [Hin84] for an introduction to equivariant Morse theory on ΛM.

Chapter 3

Rational S^1-equivariant Perfectness of the Energy Function

In the case of the canonical metric g_0 on S^3 we let $B_k \cong T^1S^3$ be the manifold of k-times iterated geodesics. The energy function is a Morse-Bott function for the metric g_0. The manifolds B_k, $k \in \mathbb{N}_{>0}$ and the point curves S^3 are the only critical manifolds for E and the induced action of $S^1/\mathbb{Z}_k$ on B_k is free. The S^1-equivariant cohomology of T^1S^3 can thus be calculated from the Gysin sequence for the bundle $S^1 \to T^1S^3 \to T^1S^3/S^1$ and is given by

$$H^i_{S^1}(T^1S^3;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = 0, 4, \\ \mathbb{Q}^2, & i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Since by [Zil77] the indices of the critical manifolds B_k in ΛS^3 are $2(2k-1)$, $k \in \mathbb{N}_{>0}$, we see that for all k the rational S^1-equivariant cohomology of B_k is concentrated in even degrees. Hence by the Lacunary Principle the energy function is perfect relative to S^3 for rational S^1-equivariant cohomology.

Proposition 3.1. *The rational S^1-equivariant cohomology of ΛS^3 relative to S^3 is given by*

$$H^i_{S^1}(\Lambda S^3, S^3; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = 2 \\ \mathbb{Q}^2, & i = 2k,\ k \in \mathbb{N},\ k > 1 \\ 0, & \textit{otherwise.} \end{cases}$$

More generally, Hingston calculated the S^1-equivariant cohomology of the free loop space of any compact rank one symmetric space; see [Hin84, Section 4.2].

Let g be a metric on S^3 all of whose geodesics are closed and normalized so that 2π is the least common period. We assume that there exist exceptional geodesics on (S^3, g) of period $2\pi/n$ for $n > 1$. Since all geodesics are closed with common period 2π the geodesic flow defines an effective and orientation preserving action of $\mathbb{R}/2\pi\mathbb{Z} = S^1$ on T^1S^3. Choose a metric on T^1S^3 such that the action of S^1 becomes isometric. The full unit tangent bundle corresponds to geodesics of period 2π and the closed geodesics of length $(k/n)2\pi$ can be identified with the fixed point set of the element $e^{2\pi i/n} \in S^1$ in T^1S^3, i.e. the fixed point set of $\mathbb{Z}_n \subseteq S^1$. The fixed point set of $\mathbb{Z}_n \subseteq S^1$ again has an effective and orientation preserving action of $S^1/\mathbb{Z}_n = S^1$ since S^1 is abelian. Since the critical sets of the energy function can be identified with the fixed point sets of some $g \in S^1$ and the metric is chosen so that the action is isometric, it follows that the critical sets are compact, totally geodesic submanifolds of T^1S^3.

We recall an observation made by Wilking, [Wil01, Proof of Step 3]: If all geodesics are closed, the energy function is a Morse-Bott function. To see this, one notes that a critical manifold $N \subseteq T^1S^3$ of geodesics of length $2\pi/n$ is a connected component of the fixed point set of an element $g = \mathrm{e}^{2\pi i/n} \in S^1$. The dimension of the critical manifold is equal to the multiplicity of the eigenvalue 1 of the map g_{*v} at a fixed point v. Since the differential of the geodesic flow is the Poincaré map, the multiplicity of the eigenvalue 1 at v is equal to the dimension of the vector space of $2\pi/n$-periodic Jacobi fields along the geodesic $c(t) = \exp(tv)$, $t \in [0, 2\pi/n]$ ($\dot{c}$ is considered as a periodic Jacobi field as well). Since the null space of $\mathrm{Hess}_c(\mathrm{E})$ consists of $2\pi/n$-periodic Jacobi vector fields, we see that the kernel of $\mathrm{Hess}_c(\mathrm{E})$ is equal to the tangent space of the critical manifolds.

We know that the geodesic flow acts orientation preserving on the fixed point sets, but we need to show that the fixed point sets are indeed orientable. This and the fact that the fixed point sets have even codimension follow from the following lemma.

Lemma 3.1. *The fixed point sets* $\mathrm{Fix}(\mathbb{Z}_n) \subseteq T^1S^3$ *have even codimension and are orientable.*

Proof. We recall a few facts about the Poincaré map; see [BTZ82, page 216]. Let c

be a closed geodesic with $c'(0) = v \neq 0$. Let φ_t denote the geodesic flow and note that a closed geodesic with $v = c'(0)$ corresponds to a periodic orbit $\varphi_t v$. The flow φ_t maps the set $T_r S^3 = \{v \in TS^3 \mid |v| = r\}$ into itself. The Poincaré map $\mathcal{P}$ of c is the return map of a local hypersurface $N \subset T_r S^3$ transversal to v. The linearized Poincaré map is up to conjugacy independent of N and is given by $P = D_v \mathcal{P}$. One can choose N such that $T_v N = V \oplus V$, where $V = v^\perp \subseteq T_p S^3$, where p is the footpoint of v. On $V \oplus V$ there is a natural symplectic structure which is preserved by P. As above we identify the critical manifolds with the fixed point sets of $\mathbb{Z}_n \subset S^1$ and notice that since the critical manifolds are nondegenerate, the dimension of the fixed point sets is equal to the multiplicity of the eigenvalue 1 of P plus one. By [Kli78, Proposition 3.2.1] we know that the multiplicity of 1 as an eigenvalue of P is even. Hence the dimension of $\mathrm{Fix}(\mathbb{Z}_n) \subseteq T^1 S^3$ is odd and the codimension is even.

If the fixed point sets are one or five dimensional, they are clearly orientable, so assume $\dim \mathrm{Fix}(\mathbb{Z}_n) = 3$. In this case the normal form of P has a 2×2 identity block and a 2×2 block which is a rotation, possibly by an angle π. Denote the two dimensional subspaces by V_{id} and V_{rot}. Using the symplectic normal form for P we know that V_{id} and V_{rot} are orthogonal with respect to the symplectic form and that the restriction of the symplectic form to each subspace is nondegenerate; see [BTZ82, page 222]. If we consider the normal bundle $\nu \,\mathrm{Fix}(\mathbb{Z}_n)$ with fiber $\nu_v \,\mathrm{Fix}(\mathbb{Z}_n)$ for $v \in \mathrm{Fix}(\mathbb{Z}_n)$, we have $\nu_v \,\mathrm{Fix}(\mathbb{Z}_n) = V_{\mathrm{rot}}$. Since V_{rot} is a symplectic subspace, it carries a canonical orientation. This gives a canonical orientation on each fiber of the normal bundle, which shows that the normal bundle is orientable and hence that $\mathrm{Fix}(\mathbb{Z}_n)$ is orientable. $\square$

The one dimensional critical manifolds are diffeomorphic to circles. If c is a one dimensional critical manifold, the geodesic $\overline{c}(t) = c(-t)$ is a second critical manifold of the same index. If the critical manifold is five dimensional it is the full unit tangent bundle. If the exceptional critical manifold that consists of prime closed geodesics is five dimensional then all geodesics are closed with period $2\pi/n$ contradicting the assumption.

We now state and prove the main result of the thesis.

Theorem 1. *Let g be a metric on S^3 all of whose geodesics are closed. Then relative to the point curves the energy function* E *is perfect with respect to rational S^1-equivariant cohomology.*

Proof. The proof takes up several pages, and we first give a short outline. We will use the Index Parity Theorem repeatedly; see [Wil01, Theorem 3]. The Index Parity Theorem states that for an oriented Riemannian manifold M^n all of whose geodesics are closed, the index of a geodesic in the free loop space is even if n is odd, and odd if n is even; in particular, in our case it states that all indices are even. The idea is to show that the contributions $H^i_{S^1}(D(N), S(N); \mathbb{Q})$ from a critical manifold N to the S^1-equivariant cohomology of ΛS^3 occur in even degrees only. If the negative bundle over N is orientable this is by the Thom Isomorphism equivalent to showing that $H^i_{S^1}(N;\mathbb{Q}) = 0$ for i odd. Perfectness of the energy function then follows from the Lacunary Principle, since the critical manifolds all have even index.

We first show that the negative bundles over the one and five dimensional critical manifolds are oriented and that the critical manifolds only have S^1-equivariant cohomology in even degrees. If the critical manifold is three dimensional we first use Smith Theory to show that the S^1-Borel construction is rationally homotopy equivalent to S^2. If the negative bundle is oriented the contributions occur in even degrees only. If the negative bundle is nonorientable we use Morse Theory and a covering space argument to show that the critical manifold does not contribute to the S^1-equivariant cohomology.

We begin the proof by considering the five dimensional case. The five dimensional critical manifold is diffeomorphic to the unit tangent bundle. It is clear that the negative bundles over the unit tangent bundle are oriented, since the unit tangent bundle is simply connected. The Borel construction $T^1S^3 \times_{S^1} ES^1$ is rationally homotopy equivalent to T^1S^3/S^1, since the action has finite isotropy groups, and thus the possible degrees where $T^1S^3 \times_{S^1} ES^1$ has nonzero rational cohomology is zero through four. Using these facts and the Gysin sequence for the bundle $S^1 \to T^1S^3 \times ES^1 \to T^1S^3 \times_{S^1} ES^1$ we see that $H^*_{S^1}(T^1S^3;\mathbb{Q}) = H^*(S^2 \times S^2;\mathbb{Q})$ and hence has nonzero classes in even degrees only.

Next, we show that the negative bundles over the one dimensional critical manifolds are oriented and thus, since the S^1-Borel construction of the one dimensional critical manifolds has rational cohomology as a point, the contributions occur in even degrees only.

Proposition 3.2. *The negative bundles over the one dimensional critical manifolds are (S^1-equivariantly) orientable.*

Proof. The negative bundle over the critical manifold consisting of a prime closed geodesics c is orientable, since we can define an orientation in one fiber of the negative bundle and use the free $S^1/\mathbb{Z}_n$-action to define an orientation in the other fibers.

Let ξ_n be the negative bundle over the n-times iterated geodesic. The representation of $\mathbb{Z}_n$ on the fibers of ξ_n is presented in [Kli78, Proposition 4.1.5]. The representation of $\mathbb{Z}_n$ is the identity on a subspace of dimension $\mathrm{Index}(c)$ (which corresponds to the image of the bundle ξ_1 under the n-times iteration map) and is a sum of two dimensional real representations given by multiplication by $\mathrm{e}^{\pm 2\pi i p/n}$ on a vector space of even dimension. If n is even, $\mathbb{Z}_n$ also acts as $-\mathrm{id}$ on a subspace of dimension $\mathrm{Index}(c^2) - \mathrm{Index}(c)$.

We know by [Kli78, Proposition 4.1.5] that the dimension of the subspace on which a generator T of $\mathbb{Z}_n$ acts as $-\mathrm{id}$ is equal to $\mathrm{Index}(c^2) - \mathrm{Index}(c)$, which by the Index Parity Theorem is even. By [Kli78, Lemma 4.1.4] the pair $(D^k/\mathbb{Z}_n, S^{k-1}/\mathbb{Z}_n)$ is orientable ($k = \mathrm{Index}(c^n)$), since the dimension of the subspace on which T acts as $-\mathrm{id}$ is even dimensional. Pick an orientation in one fiber of the negative bundle invariant under the action of $\mathbb{Z}_n$. Use the $S^1/\mathbb{Z}_n$- action to define an orientation in any other fiber of the negative bundle. Since the orientation is chosen to be invariant under the action of $\mathbb{Z}_n$ the orientation on the negative bundle is well-defined. $\square$

We now treat the three dimensional case and first prove the following important fact.

Proposition 3.3. *Assume that the critical manifold N has dimension three. Then the S^1-Borel construction of N is rationally homotopy equivalent to S^2.*

Proof. Assume that $\mathbb{Z}_n$ acts trivially on N. We start by considering the quotient $N \to N/(S^1/\mathbb{Z}_n)$. The action of $S^1/\mathbb{Z}_n$ is effective and has isotropy at a closed geodesic which is an lth iterate, for some l, of a closed geodesic in a one dimensional critical manifold. The $S^1/\mathbb{Z}_n$-action normal to the $S^1/\mathbb{Z}_n$-orbit acts as a rotation by $e^{2\pi i/l}$. Hence $N/(S^1/\mathbb{Z}_n)$ is a two dimensional orbifold, since a neighborhood of an arbitrary point in $N/(S^1/\mathbb{Z}_n)$ is homeomorphic to $\mathbb{R}^2/\mathbb{Z}_l$. However, the quotient $\mathbb{R}^2/\mathbb{Z}_l$ is homeomorphic to $\mathbb{R}^2$ so in particular $N/(S^1/\mathbb{Z}_n)$ is homeomorphic to a surface. Since by Lemma 3.1 $\mathrm{Fix}(\mathbb{Z}_n)$ is orientable and the action of $S^1/\mathbb{Z}_n$ is orientation preserving, the quotient is an orientable surface of genus g.

Using the Gysin sequence we can calculate $H^1(N;\mathbb{Q})$ from the bundle $S^1 \to N \times ES^1 \to N \times_{S^1} ES^1$, since we know that $N/(S^1/\mathbb{Z}_n)$ is rationally homotopy equivalent to $N \times_{S^1} ES^1$. The Gysin sequence yields

$$H^i(N;\mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 3 \\ \mathbb{Q}^{2g}, & i = 1, 2 \text{ if } \chi \neq 0, \\ \mathbb{Q}^{2g+1}, & i = 1, 2 \text{ if } \chi = 0, \end{cases}$$

where χ denotes the Euler class of the bundle.

If n is a prime, p, we can use Smith theory for the $\mathbb{Z}_n$-action on T^1S^3 to bound the sum of the Betti numbers of the fixed point sets, i.e. we know that the sum $\sum b_i(\mathrm{Fix}(\mathbb{Z}_p);\mathbb{Z}_p) \leq \sum b_i(T^1S^3;\mathbb{Z}_p) = 4$ for an arbitrary prime p; see [Bre72, Theorem 4.1]. Hence we have $\sum b_i(\mathrm{Fix}(\mathbb{Z}_p);\mathbb{Q}) \leq 4$, by the Universal Coefficient Theorem. If n is not prime we choose a p such that $p|n$ and such that $\mathrm{Fix}(\mathbb{Z}_p)$ is three dimensional, which is possible since N is three dimensional and the only critical manifold of dimension five is T^1S^3. Then $N \subseteq \mathrm{Fix}(\mathbb{Z}_p)$ and since both N and $\mathrm{Fix}(\mathbb{Z}_p)$ are closed three dimensional submanifolds of T^1S^3, N equals a component of $\mathrm{Fix}(\mathbb{Z}_p)$. Hence we have $\sum b_i(N;\mathbb{Q}) \leq 4$, which implies that $g = 0$. $\square$

Thus, if the negative bundle is orientable we only have contributions in even degrees. We now treat the case where the negative bundle is nonorientable.

Proposition 3.4. *Let $\xi^- \to N$ be a nonorientable negative bundle over a three dimensional critical manifold N. Then the cohomology groups $H^*_{S^1}(D(N), S(N);\mathbb{Q})$ vanish.*

Proof. Let $p\colon \tilde{N} \to N$ be the twofold cover such that the pull-back $p^*\xi^- = \tilde{\xi}^-$ is orientable. Let ι be the covering involution on $\tilde{N}$ which lifts to $\tilde{\xi}^-$. Note that the S^1-equivariant negative bundle $\xi^-_{S^1} \to N_{S^1}$ is also nonorientable. Lift the action of $S^1/\mathbb{Z}_n$, or possibly a twofold cover of $S^1/\mathbb{Z}_n$, on ξ^- to an action on $\tilde{\xi}^-$ such that ι becomes equivariant with respect to this action. Let $\tilde{\xi}^-_{S^1} \to \tilde{N}_{S^1}$ be the oriented twofold cover of $\xi_{S^1} \to N_{S^1}$. We know from Proposition 3.3 that N_{S^1} is rationally homotopy equivalent to S^2. The action of $S^1/\mathbb{Z}_n$ on $\tilde{N}$ also has finite isotropy groups, so by an argument similar to the one in the proof of Proposition 3.3 we see that $\tilde{N}/(S^1/\mathbb{Z}_n)$ is an oriented surface of genus g, F_g. The Borel construction $\tilde{N}_{S^1}$ is rationally homotopy equivalent to F_g.

There are two cases to consider: $g = 0$ and $g > 0$. We first treat the case $g = 0$. Since the bundle $\tilde{\xi}^-_{S^1}$ is orientable there exists a Thom class and this cohomology class changes sign under the action of ι^*, since otherwise it would descend to give a Thom class for $\xi^-_{S^1}$ making it orientable. By general covering space theory we know that $H^*_{S^1}(D(\tilde{N}), S(\tilde{N}); \mathbb{Q})^{\iota^*} = H^*_{S^1}(D(N), S(N); \mathbb{Q})$ and $H^*_{S^1}(\tilde{N}; \mathbb{Q})^{\iota^*} = H^*_{S^1}(N; \mathbb{Q})$, where the superscript ι^* denotes the classes that are fixed under the action of ι^*, and where as before, $D(N) = D(\xi(N))$ etc.

By the Thom isomorphism we have $H^i_{S^1}(D(\tilde{N}), S(\tilde{N}); \mathbb{Q}) = H^{i-\lambda(N)}_{S^1}(\tilde{N}; \mathbb{Q})$ and since $g = 0$ we have $H^i_{S^1}(\tilde{N}; \mathbb{Q}) = 0$ for all $i \neq 0, 2$, which means that we only have to see what happens to the degree zero and degree two cocycles. By the above, this implies that $H^i_{S^1}(D(N), S(N); \mathbb{Q}) = 0$ for $i \neq \lambda(N), \lambda(N) + 2$. The action of ι^* on $H^0_{S^1}(\tilde{N}; \mathbb{Q})$ is trivial and since the Thom class changes sign, the action of ι^* on $H^{\lambda(N)}_{S^1}(D(\tilde{N}), S(\tilde{N}); \mathbb{Q})$ has no fixed points. Hence we conclude that $H^{\lambda(N)}_{S^1}(D(N), S(N); \mathbb{Q}) = 0$.

Since both N and $\tilde{N}$ are two spheres we have $H^2_{S^1}(\tilde{N}; \mathbb{Q}) = \mathbb{Q}$ and $H^2_{S^1}(N; \mathbb{Q}) = \mathbb{Q}$, but also that $H^2_{S^1}(\tilde{N}); \mathbb{Q})^{\iota^*} = H^2_{S^1}(N; \mathbb{Q}) = \mathbb{Q}$. This implies that ι^* acts trivially on $H^2_{S^1}(\tilde{N};\mathbb{Q})$ and, since the Thom class changes sign under the action of ι^*, as $-\mathrm{id}$ on $H^{\lambda(N)+2}_{S^1}(D(\tilde{N}),S(\tilde{N});\mathbb{Q})$. We conclude that $H^{\lambda(N)+2}_{S^1}(D(N),S(N);\mathbb{Q})=0$.

The second case to consider is $g>0$. In that case we will derive a contradiction. Let $2k$ be the minimal index of a critical manifold on which $\mathbb{Z}_n$ acts trivially and whose negative bundle is nonorientable. If there is more than one critical manifold of index $2k$ with nonorientable negative bundle, we repeat the argument for each manifold. If the action of $S^1/\mathbb{Z}_n$ had been free, N_{S^1} would be homotopy equivalent to S^2 and the negative bundle would have been orientable. Hence the action of $S^1/\mathbb{Z}_n$ on N has fixed points and since the action is effective, the fixed points are one dimensional critical manifolds.

Since the map $\tilde{N}/S^1 \to N/S^1$ is a branched covering, the Riemann-Hurwitz formula implies that there are $2g+2$ branched points. Exceptional orbits of the S^1-action on $\tilde{N}$ are circles, since the action is effective, and they project down to exceptional orbits (shorter geodesics) for the action on N. Branched points for the covering $\tilde{N}/S^1 \to N/S^1$ correspond to orbits where the isotropy of the action on N is bigger than for the action on $\tilde{N}$. Hence we see that the branched points come from exceptional orbits for the action on N and hence that there are at least $2g+2$ exceptional orbits in N.

The exceptional orbits are shorter geodesics, and since by the Bott Iteration Formula $\mathrm{Index}(c^q) \geq \mathrm{Index}(c)$, these circles must all have index less than or equal to $2k$. By the following lemma we deduce that the index must be equal to $2k$. For later reference we state this lemma separately.

Lemma 3.2. *There exists no one dimensional critical manifolds of index less than $2k$.*

Proof. Assume for contradiction that there exists a one dimensional critical manifold of index $2h < 2k$, $h \geq 2$. This critical manifold hence contributes a $\mathbb{Q}^2$ in degree $2h$. A three dimensional critical manifold contributes a $\mathbb{Q}$ in the degree equal to the index and a $\mathbb{Q}$ in degree equal to the index plus 2, since the negative bundle is orientable (hence a total of two classes). The unit tangent bundle contributes a total of 4 classes. There are no cancellations in degree less than $2k$, since the contributions all have even degrees. The total number of classes needed to "fill the gaps" (from $H^2_{S^1}(\Lambda S^3, S^3; \mathbb{Q})$ to $H^{2h-2}_{S^1}(\Lambda S^3, S^3; \mathbb{Q})$) in the cohomology is $2h-3$, since $H^2_{S^1}(\Lambda S^3, S^3; \mathbb{Q}) = \mathbb{Q}$. Notice that as $2h < 2k$ no cancellations are possible, since all contributions occur in even degrees. This implies that no three dimensional critical manifold can have index $2h-2$ and the unit tangent bundle cannot have index $2h-2$ or $2h-4$, since otherwise $\dim H^{2h}_{S^1}(\Lambda S^3, S^3; \mathbb{Q}) \geq 3$, since we have contributions from the two circles and the three or five dimensional critical manifold. Hence we have to fill an odd number of holes with contributions that only come in pairs or quartets. This is clearly impossible.

If $h = 1$ we get a contribution of $\mathbb{Q}^2$ in degree 2, which cannot cancel out, since the index two critical manifold only contributes in even degrees. This contradicts the fact that $H^2_{S^1}(\Lambda S^3, S^3; \mathbb{Q}) = \mathbb{Q}$. □

By a similar argument, we see that there is at most one three or five dimensional critical manifold for each even index less than $2k$. If the unit tangent bundle has index $2j$, there is no critical manifold of index $2j+2$, since $H^2_{S^1}(T^1S^3;\mathbb{Q}) = \mathbb{Q}^2$.

Now we derive the contradiction. This is done by considering the possible contributions to $H^{2k}_{S^1}(\Lambda S^3, S^3;\mathbb{Q})$. The three dimensional critical manifold of index $2k-2$ contributes a $\mathbb{Q}$ in degree $2k$, a five dimensional critical manifold of index $2k-4$ contributes a $\mathbb{Q}$ in degree $2k$, and similarly a five dimensional critical manifold of index $2k-2$ contributes a $\mathbb{Q}^2$ in degree $2k$. Hence we get a contribution to $H^{2k}_{S^1}(\Lambda S^3, S^3;\mathbb{Q})$ of at least a $\mathbb{Q}$ from the critical manifold of index $2k-2$ or $2k-4$ and a $\mathbb{Q}^{2g+2}$ from the exceptional orbits. By the Thom isomorphism we have $H^{2k+1}_{S^1}(D(\tilde{N}), S(\tilde{N});\mathbb{Q}) = H^1_{S^1}(\tilde{N};\mathbb{Q})$ and furthermore we have by covering space theory that $H^{2k+1}_{S^1}(D(\tilde{N}), S(\tilde{N});\mathbb{Q})^{\iota^*} = H^{2k+1}_{S^1}(D(N), S(N);\mathbb{Q})$, which implies that $H^1_{S^1}(\tilde{N};\mathbb{Q})^{\iota^*} = H^{2k+1}_{S^1}(D(N), S(N);\mathbb{Q})$. Since $H^1_{S^1}(\tilde{N};\mathbb{Q}) = \mathbb{Q}^{2g}$, we see that the maximal contribution to $H^{2k+1}_{S^1}(\Lambda S^3, S^3;\mathbb{Q})$ is $\mathbb{Q}^{2g}$, which for example happens if ι^* acts as $-\mathrm{id}$ on $\mathbb{Q}^{2g}$. This yields a contradiction since we now have $\dim H^{2k}_{S^1}(\Lambda S^3, S^3;\mathbb{Q}) \geq 3$.

If there exists another critical manifold of index $2k$ with nonorientable negative bundle we repeat the argument above. The exceptional geodesics that contribute in degree $2k$ are distinct from the ones in other critical manifolds since the iterates of the shorter geodesics lie in different connected critical manifolds. $\square$

This finishes the proof that E is perfect. $\square$

A consequence of the perfectness of the energy function is the following general fact.

Corollary 3.1. *The minimal index of a critical manifold consisting of nonconstant geodesics is two. For every number $2k$ there exists at most one connected critical manifold of index $2k$ for every $k \geq 1$.*

Proof. As a global minimum of E, S^3 has index zero. If one of the critical manifolds N consisting of nonconstant geodesics has index zero, $H^0_{S^1}(\Lambda S^3; \mathbb{Q})$ would be at least two dimensional, which is not the case. If the minimal index, $\mathrm{Index}(N) = 2i$, $i > 1$, then $H^2(\Lambda S^3; \mathbb{Q}) = \cdots = H^{2i-1}(\Lambda S^3; \mathbb{Q}) = 0$, which is not the case. That there is at most one critical manifold of a given index is clear by an argument similar to the one in the proof of Lemma 3.2. $\square$

Chapter 4

Orientability of Negative Bundles

We want to use the fact that E is perfect to conclude that all negative bundles over the three dimensional critical manifolds are indeed orientable. Notice that this is not a circular argument, since the proof of perfectness does not use orientability.

Proposition 4.1. *There do not exist any one dimensional critical manifolds.*

Proof. This is similar to the argument in the proof of Lemma 3.2. By S^1-equivariant perfectness of E there can be at most two circles of index $2j$. Let $2k$, $k \geq 2$ be the minimal index of a one dimensional critical manifold. This critical manifold contributes a $\mathbb{Q}^2$ in degree $2k$. We must then fill the $2k-3$ gaps from $H^2_{S^1}(\Lambda S^3, S^3; \mathbb{Q})$ to $H^{2k-2}_{S^1}(\Lambda S^3, S^3; \mathbb{Q})$ with contributions that come in pairs or quartets, clearly impossible. If $k = 1$ we get a contradiction since $H^2_{S^1}(\Lambda S^3, S^3; \mathbb{Q}) = \mathbb{Q}$ □

Corollary 4.1. *The negative bundles over the three dimensional critical manifolds are (S^1-equivariantly) orientable.*

Proof. First note that if the dimension of a critical manifold N is three, the action of $S^1/\mathbb{Z}_n$ on N is free, since there are no one dimensional critical manifolds.

Hence the S^1-Borel construction $N \times_{S^1} ES^1$ is homotopy equivalent to $N/(S^1/\mathbb{Z}_n)$ and by the proof of Proposition 3.3 $N/(S^1/\mathbb{Z}_n)$ is homotopy equivalent to S^2. Since S^2 is simply connected the bundle is orientable. We conclude that the negative bundles are also oriented as ordinary vector bundles. $\square$

It was shown in the previous chapter that the negative bundles over the one and five dimensional critical manifolds are orientable, so this corollary finishes the proof that all negative bundles are orientable.

Chapter 5

Topology of the Three Dimensional Critical Manifolds

By calculating the Euler class of the bundle $S^1 \to N \to N/S^1 = S^2$ we will deduce that the three dimensional critical manifolds are integral cohomology three spheres or lens spaces.

Theorem 2. *Assume that N is a three dimensional critical manifold. The quotient $N/S^1 = S^2$ is endowed with a symplectic structure which corresponds to the Euler class of the S^1-bundle $S^1 \to N \to N/S^1$; in particular the Euler class is nonzero and N has the integral cohomology of either the three sphere or a lens space.*

Proof. Note that by Corollary 3.1 N is connected. Parts of the proof rely on an argument in [Bes78]; see [Bes78, Definition 1.23, Proposition 2.11 and 2.16]. We will describe some extra structure on the fixed point set N. In [Bes78, Chapter

2] the author considers a manifold all of whose geodesics are closed with the same least period 2π. In that case the action of S^1 on T^1S^3 is free and one gets a principal S^1 bundle $p\colon T^1S^3 \to T^1S^3/S^1$. There is a canonical connection $\alpha \in H^1_{\mathrm{dR}}(T^1S^3; \mathcal{L}(S^1))$ on T^1S^3 constructed as follows: Let $p_{TS^3}\colon TTS^3 \to TS^3$ be the projection and $T_{p_{S^3}}\colon TTS^3 \to TS^3$ be the tangent map. For $X \in TTS^3$ we define $\tilde{\alpha}(X) = g(T_{p_{S^3}}(X), p_{TS^3}(X))$. This form is the pullback to the tangent bundle of the canonical one form on the cotangent bundle. Define a horizontal distribution on TT^1S^3 by $Q_u = \{X \in T_uT^1S^3 \mid \alpha(X) = 0\}$. Then $T_uT^1S^3 = \mathbb{R}Z \oplus Q_u$, where Z is the geodesic vector field on TT^1S^3, and the corresponding connection form is $\tilde{\alpha}$ restricted to T^1S^3 which we denote by α. The Lie algebra of S^1 is abelian and $d\alpha$ is horizontal, so the curvature form of α is $d\alpha$. Since the two form $d\alpha$ is invariant under the action of S^1, it is basic. The form $d\alpha$ is also the restriction to the unit tangent bundle of the pullback to the tangent bundle of the canonical two form on the cotangent bundle. Hence $d\alpha$ is nonzero on the complement of Z on every $T_uT^1S^3$, $u \in T^1S^3$. By Chern-Weil Theory there exists $\omega \in H^2_{\mathrm{dR}}(T^1S^3/S^1)$ such that $p^*(\omega) = d\alpha$. By [KN69, Theorem 5.1] this class is the Euler class of the bundle. By [Bes78, Proposition 2.11] the class ω is a symplectic form on T^1S^3/S^1 (ω being degenerate would mean that $d\alpha = 0$ on a nonempty subset of the complement of Z, which is not the case). Hence the Euler class is nonzero.

We now carry the argument over to the three dimensional critical manifolds. By Proposition 4.1 we know that the action of S^1 on N is free and that N/S^1 is homeomorphic to S^2. Consider the principal bundle $S^1 \to N \to S^2$, with projection q. We want to conclude that the Euler class of this bundle is nonzero. First note that the geodesic vector field is tangent to the fixed point set. Consider the restriction of α to TN and define as above a distribution on TN by $\tilde{Q}_u = \{X \in T_u N \mid \alpha(X) = 0\}$. Then we have $TN = \mathbb{R}Z \oplus \tilde{Q}_u$, since Z is a vector field on TN with $\alpha(Z) = 1$; see [Bes78, 1.57]. The restriction of α to TN is invariant under the action of S^1 given by the geodesic flow, since by [Bes78, 1.56] $L_Z\alpha = 0$, so the distribution $\tilde{Q}$ is invariant under the action of S^1. Since the map $u \mapsto \tilde{Q}_u$ is clearly smooth, $\tilde{Q}$ is a horizontal distribution and since $\alpha(Z) = 1$, α is the connection of the distribution. Similarly to the above, we see that since the Lie algebra of S^1 is abelian the curvature of the bundle is $d\alpha$ ($d\alpha$ is horizontal since $d\alpha(Z, -) = 0$ by [Bes78, 1.56]). As $d\alpha$ is horizontal and invariant under the action of S^1 ($L_Z d\alpha = 0$ by [Bes78, 1.57]), it is basic. Hence we can find a form $\omega \in H^2_{\mathrm{dR}}(S^2)$ such that $q^*(\omega) = d\alpha$. Thus, to see that the quotient is symplectic, it suffices to show that the curvature is nonzero on $\tilde{Q}$. This follows from the following general statement about symplectic reduction. The proof of this statement is that, similarly to the proof of Lemma 3.1, the $+1$ eigenspace of the Poincaré map is a symplectic subspace.

Lemma 5.1. *Let (V^4, τ) be a symplectic vector space and let $\mathbb{Z}_n$ act on V by linear symplectic transformations. Assume that* $\dim \mathrm{Fix}(\mathbb{Z}_n) = 2$. *Then* $\mathrm{Fix}(\mathbb{Z}_n)$ *is a symplectic subspace.*

Consider the tangent space to T^1S^3 at a point $u \in T^1S^3$. This splits as a direct sum $Z \oplus Q_u$ and $d\alpha \neq 0$ on Q_u and on the four dimensional subspace the form $d\alpha$ is a symplectic two form and the differential of the geodesic flow acts by symplectic linear transformations. The tangent space $T_u N$ also splits as the direct sum of $Z \oplus \tilde{Q}_u$, $\tilde{Q}_u \subset T_u N$. It follows from the above lemma that the form $d\alpha$ restricted to $T_u N$ is nonzero and hence, as above, that the form $\omega \in H^2_{\mathrm{dR}}(S^2)$ makes the quotient into a symplectic manifold. Again by [KN69, Theorem 5.1] the class ω is the Euler class of the S^1-bundle $S^1 \to N \to S^2$. Since the Euler class is nonzero an application of the Gysin sequence for the bundle $S^1 \to N \to S^2$ shows that N is either an integral cohomology three sphere or lens space. □

Corollary 5.1. *There are no three dimensional critical manifolds with the integral cohomology of S^3 or a lens space $S^3/\mathbb{Z}_r$, r odd.*

Proof. By Corollary 3.1 there exists at most one connected, critical manifold, N, of a given index. Let n be the multiplicity of a geodesic in N, i.e. c has length $2\pi/n$. The action of $O(2)$ leaves N invariant and hence induces an action on N which is effectively free since there are no one dimensional critical manifolds. Let $\mathbb{Z}_n \subseteq S^1$ be the ineffective kernel of the action. By identifying $S^1/\mathbb{Z}_n$ with S^1 we see that $O(2)/\mathbb{Z}_n \cong O(2)$. The group $O(2)/\mathbb{Z}_n$ acts freely on N, so in particular $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq O(2)/\mathbb{Z}_n$ acts freely on N. By a a theorem of Smith [Bre72, Theorem 8.1] $\mathbb{Z}_2$-cohomology spheres do not support a free action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, so since $S^3/\mathbb{Z}_r$, r odd, are $\mathbb{Z}_2$-cohomology spheres we have finished the proof. □

Chapter 6

Index Growth

In this section we will use the results of [BTZ82] concerning the Bott Iteration Formula to obtain complete information about the index growth of geodesics in the three and five dimensional critical manifolds. The Bott Iteration Formula states that $\mathrm{Index}(c^q) = \sum_{z^q=1} I(\theta)$, where $I\colon S^1 \to \mathbb{R}$ is a locally constant function, which will be determined in the following. We will write $I(\theta)$ for $I(z)$, $z = \mathrm{e}^{i\theta}$ and $\theta \in [0, 2\pi]$. Notice that since the critical manifolds are nondegenerate in the sense of Bott, a geodesic in a five dimensional critical manifold has Poincaré map equal to the identity, and the Poincaré map of a geodesic in a three dimensional critical manifold has a two by two identity block.

Since there exists a common period for the geodesics, we know that the Poincaré map P of a closed geodesic is a root of unity. Let c be a closed geodesic of multiplicity n. If there exists an eigenvalue $\mathrm{e}^{2\pi i/k}$ where $k|n$, the map P^k has two 2×2 identity blocks and since c^k, $k < n$, still lies in a three dimensional critical manifold this contradicts the fact that all three dimensional critical manifolds are nondegenerate. Hence the eigenvalues of P are 1 and $\mathrm{e}^{2\pi i/n}$.

The splitting numbers $S^{\pm}(\theta)$ are given by $S^{\pm}(\theta) = \lim_{\rho\to 0^{\pm}} I(\theta+\rho) - I(\theta)$, for $0 \le \theta \le \pi$, and completely determine the function I, since we know that $I(\theta) = I(2\pi - \theta)$, I is locally constant and that I only jumps at eigenvalues of P, i.e. at a value of θ where $S^{\pm}$ is nonzero. If with respect to a symplectic basis the Jordan normal form of the Poincaré map contains a 2×2 block of the form

$$\begin{bmatrix} \cos(\theta) & \sigma\sin(\theta) \\ -\sigma\sin(\theta) & \cos(\theta) \end{bmatrix}$$

for $\sigma \in \{\pm 1\}$ and $0 < \theta < \pi$, we say that the sign of the Jordan block is σ.

We now calculate the splitting numbers $S^+(0)$, $S^-(0)$, $S^+(2\pi/n)$ and $S^-(2\pi/n)$. We note that in the case of a five dimensional critical manifold there is only one eigenvalue 1 and the splitting numbers corresponding to that eigenvalue are 2.

Lemma 6.1. *Let n be the multiplicity of a closed geodesic c in a three dimensional critical manifold and let P be its Poincaré map. The splitting numbers of P, $S^+(0)$ and $S^-(0)$ are both equal to 1. The splitting numbers $S^+(2\pi/n)$ and $S^-(2\pi/n)$ depend on the sign σ of the Jordan block and for $\sigma = +1$ we have $S^+(2\pi/n) = 1$, $S^-(2\pi/n) = 0$ and for $\sigma = -1$ we have $S^+(2\pi/n) = 0$, $S^-(2\pi/n) = 1$. If $n = 2$ the splitting numbers are all equal to 1.*

Proof. This follows directly from [BTZ82, Theorem 2.13] and the calculation of the

Jordan blocks, [BTZ82, page 222]. □

Next we want to determine the function I, which is quite simple knowing the splitting numbers.

Proposition 6.1. *Let c be a geodesic of index l in a three dimensional critical manifold. The function $I\colon S^1 \to \mathbb{R}$ is for $\sigma = 1$ given by*

$$I(\theta) = \begin{cases} l, & \text{for } \theta = 0, \\ l+1 & \text{for } \theta \in (0, 2\pi/n], \\ l+2 & \text{for } \theta \in (2\pi/n, \pi]. \end{cases}$$

If $\sigma = -1$ or $n = 2$, I is given by

$$I(\theta) = \begin{cases} l, & \text{for } \theta = 0, \\ l+1 & \text{for } \theta \in (0, 2\pi/n), \\ l & \text{for } \theta \in [2\pi/n, \pi]. \end{cases}$$

Proof. This follows directly from the previous lemma. □

Remark. The index of the iterate c^k, $k = 2, 3, \ldots$, of a geodesic c in a three dimensional critical manifold of multiplicity n and some given index can now be calculated from Proposition 6.1 and the Bott Iteration Formula $\text{Index}(c^q) = \sum_{z^q=1} I(\theta)$. In the case of a five dimensional critical manifold, 1 is the only one eigenvalue and the corresponding splitting numbers are 2.

Chapter 7

Reductions of the Berger Conjecture in Dimension Three

Note that by Chapter 4 the negative bundles are oriented. The first consequence of the preceding chapters is the following theorem.

Theorem 3. *Let g be a metric on S^3 all of whose geodesics are closed. The geodesics have the same least period if and only if the energy function is perfect for ordinary cohomology.*

Proof. By [Zil77] the cohomology ring $H^*(\Lambda S^3; \mathbb{Z})$ contains no torsion, so if E is perfect Corollary 5.1 implies that T^1S^3 is the only critical manifold and hence that all geodesics are closed of the same least period. If all geodesics are closed of the same least period the only critical manifold is T^1S^3. Theorem 1 implies that the indices must be $2(2k-1)$, $k \geq 1$, which implies that E is perfect. $\square$

For reductions of the conjecture, we first consider the case where $\mathrm{Index}(T^1S^3) = 2$. If the action of S^1 has an isotropy subgroup $\mathbb{Z}_q$ there exists a three dimensional critical manifold $\mathrm{Fix}(\mathbb{Z}_q) \subset T^1S^3$. Since by the Bott Iteration Formula we have $\mathrm{Index}(c^q) \geq \mathrm{Index}(c)$ we see that the three dimensional critical manifold must also have index two, since the minimal index of a critical manifold consisting of nonconstant geodesics is two, by Corollary 3.1. This is a contradiction since there exists at most one critical manifold of a given index, by Corollary 3.1.

Next let c be a geodesic of index two which lies in a three dimensional critical manifold N and such that c has length $2\pi/n$. We have three cases to consider: Either $n = 2$ or $n > 2$ and the sign σ of the nontrivial Jordan block of N is ± 1. By the Bott Iteration Formula we have $\mathrm{Index}(c^2) = I(\pi) + \mathrm{Index}(c)$. Hence it follows from Proposition 6.1 that if $n = 2$ or $n > 2$ and $\sigma = -1$ that $\mathrm{Index}(c^2) = 4$. Furthermore, if $\sigma = +1$, $\mathrm{Index}(c^2) = 6$.

Now we assume that the sectional curvature K satisfies $a/4 \leq K \leq a$. It was proved in [BTZ83] that

$$H_*(\Lambda^{16\pi^2/2a}S^3, \Lambda^{4\pi^2/2a}S^3; \mathbb{Z}) = H_{*-2}(T^1S^3; \mathbb{Z}), \tag{7.1}$$

which has a $\mathbb{Z}$ in degree $2, 4, 5$ and 7. By the curvature assumption and since S^3 is simply connected, we know that the injectivity radius is greater than or equal to $\pi/\sqrt{a}$. Hence there exist no closed geodesics of length less than $2\pi/\sqrt{a}$ and hence of energy less than $4\pi^2/2a$. Also note that c^2 does not lie in $\Lambda^{16\pi^2/2a}S^3$. Indeed, if $\mathrm{E}(c) = 4\pi^2/2a$ and hence $\mathrm{E}(c^2) = 16\pi^2/2a$ a theorem of Tsukamoto implies that the metric g has constant sectional curvature; see [Tsu66]. This contradicts our assumption that there exist geodesics that are shorter than the common period 2π. Hence by Equation 7.1 the geodesics in N must lie in $(\Lambda^{16\pi^2/2a}S^3, \Lambda^{4\pi^2/2a}S^3)$ since by Corollary 3.1 N is the only critical manifold of index two.

Next we consider the case where $\mathrm{Index}(c^2) = 4$. By Theorem 2 we know that N contributes a rational class in degree two and five. Hence by Equation 7.1 there must exist another simple closed geodesic of index four in $\Lambda^{16\pi^2/2a}S^3$. Since $\mathrm{Index}(c^2) = 4$ we have two distinct critical manifolds of index four, contradicting Corollary 3.1.

We summarize in a theorem.

Theorem 4. *Let g be a metric on S^3 all of whose geodesics are closed with sectional curvature K satisfying $a/4 \leq K \leq a$. Then all geodesics have the same least period unless the unique closed geodesic $c \in (\Lambda^{16\pi^2/2a}S^3, \Lambda^{4\pi^2/2a}S^3)$ of index two has* $\mathrm{Index}(c^2) = 6$*, i.e. the Poincaré map of c contains a rotation with sign $\sigma = +1$*

Remark. If $\mathrm{Index}(c^2) = 6$ we conclude from Equation 7.1 that there must exist a geodesic d of index four, which must lie in a three dimensional critical manifold M, since otherwise $H^6_{S^1}(\Lambda S^3, S^3; \mathbb{Q}) = \mathbb{Q}^3$. Assume that the length of d is $2\pi/m$. If we iterate the geodesic c a multiple of n times we land in T^1S^3. The index of c^k, $k = 2, 3, \ldots$ of c is given by Proposition 6.1. When calculating the index of d^k, $k = 2, 3, \ldots$, of the geodesic d there are two cases to consider: $\sigma = \pm 1$. For some combinations of n and m we are able to derive a contradiction by showing that two distinct critical manifolds must contribute in the same degree.

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